

Relational causality and classical probability: Grounding quantum phenomenology in a superclassical theory

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Abstract

By introducing the concepts of “superclassicality” and “relational causality”, it is shown here that the velocity field emerging from an n -slit system can be calculated as an average classical velocity field with suitable weightings per channel. No deviation from classical probability theory is necessary in order to arrive at the resulting probability distributions. In addition, we can directly show that when translating the thus obtained expression for said velocity field into a more familiar quantum language, one immediately derives the basic postulate of the de Broglie-Bohm theory, i.e. the guidance equation, and, as a corollary, the exact expression for the quantum mechanical probability density current. Some other direct consequences of this result will be discussed, such as an explanation of Born’s rule and Sorkin’s first and higher order sum rules, respectively.

Keywords: quantum mechanics, entanglement, interferometry, zero-point field

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1. INTRODUCTION

In 1951, Richard Feynman published a paper [1] claiming that classical probability theory was not applicable for the description of quantum phenomena, but that instead separate “laws of probabilities of quantum mechanics” were required. In describing the propagations of electrons from a source S to a location X on the screen in a double-slit experiment, Feynman wrote: “We might at first suppose (since the electrons behave as particles) that

I. Each electron which passes from S to X must go either through hole 1 or hole 2. As a consequence of I we expect that:

II. The chance of arrival at X should be the sum of two parts, P_1 , the chance of arrival coming through hole 1, plus P_2 , the chance of arrival coming through hole 2.” However, we apparently know from experiment that this is not so (Fig. 1). Feynman concludes: “Hence experiment tells us definitely that $P \neq P_1 + P_2$ or that II is false. (...) We must conclude that when both holes are open it is not true that the particle goes through one hole or the other.”

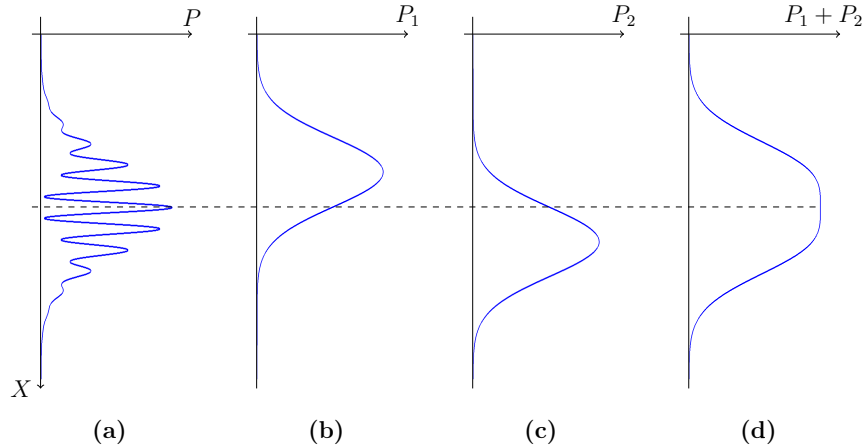


Figure 1.1: Scheme of interference at a double slit: probability distributions for (a) both slits open: P , (b) slit 2 closed: P_1 , (c) slit 1 closed: P_2 , (d) the sum of (b) and (c): $P_1 + P_2$.

However, it was shown very clearly in several later papers, notably by B. O. Koopman in 1955 [2] and by L. E. Ballentine in 1986 [3], that Feynman’s reasoning was not conclusive. In fact, Ballentine concluded that Feynman’s “argument draws its radical conclusion from an incorrect application of probability theory”, as one has to use conditional probabilities to account for the different experimental contexts: $P_1(X|C_1)$, $P_2(X|C_2)$, $P(X|C_3)$, where

the contexts C_1 , C_2 and C_3 refer to left, right and both slit(s) open, respectively. Then, it is clear from the experimental data that $P(X|C_3) \neq P_1(X|C_1) + P_2(X|C_2)$. Accordingly, Ballentine also proved that “the quantum mechanical superposition principle for amplitudes is in no way incompatible with the formalism of probability theory” [3]. Moreover, as one can easily show with the inclusion of the wave picture for electrons (see Eq. (1.3) below), one correctly obtains that $P(X|C_3) = P_1(X|C_3) + P_2(X|C_3)$.

Note that the focus purely on context-free “particles” in quantum mechanics has been the source of much confusion. We recall the clear words of John Bell in this regard: “While the founding fathers agonized over the question ‘particle’ or ‘wave’, de Broglie in 1925 proposed the obvious answer ‘particle’ and ‘wave’. Is it not clear from the smallness of the scintillation on the screen that we have to do with a particle? And is it not clear, from the diffraction and interference patterns, that the motion of the particle is directed by a wave? De Broglie showed in detail how the motion of a particle, passing through just one of two holes in a screen, could be influenced by waves propagating through both holes. And so influenced that the particle does not go where the waves cancel out, but is attracted to where they cooperate. This idea seems to me so natural and simple, to resolve the wave-particle dilemma in such a clear and ordinary way, that it is a great mystery to me that it was so generally ignored.” [4] Particularly since the impressive Paris experiments in the classical domain performed by Yves Couder, Emmanuel Fort and co-workers [5–7], it is very suggestive that some sort of combination of “particle”- and wave-mechanics is at work also in the quantum domain. Essentially, this amounts to obtaining particle distributions from calculating classical-like wave intensity distributions.

With wave-field intensities P_i for each slit i given by the squared amplitude R_i , i.e. with $P_i = R_i^2$, and with the phase difference φ , one obtains in the classical double slit scenario for the intensity after slit 1

$$P(1) = P_1 + R_1 R_2 \cos \varphi \quad (1.1)$$

and for the intensity after slit 2

$$P(2) = P_2 + R_2 R_1 \cos \varphi \quad (1.2)$$

where the first expression on the r.h.s. refers to the intensity from the slit *per se* and the second expression refers to interference with the other slit, respectively. This provides the

total intensity as

$$P = P(1) + P(2) = P_1 + P_2 + 2R_1R_2 \cos \varphi. \quad (1.3)$$

In the following we try to shed more light on these issues by combining results from the new field of “Emergent Quantum Mechanics” [8] with concepts of systems theory which we denote as “relational causality” [9]. Since the physics of different scales is concerned, like, e.g., sub-quantum and classical macro physics, we denote our sub-quantum theory as “super-classical”. (Both relational causality and superclassicality are to be defined more specifically below in Section 2.) We consider the quantum itself as an emergent system understood as off-equilibrium steady state oscillation maintained by a constant throughput of energy provided by the (“classical”) zero-point energy field. Starting with this concept, our group was able to assess phenomena of standard quantum mechanics like Gaussian dispersion of wave packets, superposition, double slit interference, Planck’s energy relation, or the Schrödinger equation, respectively, as the emergent property of an underlying sub-structure of the vacuum combined with diffusion processes reflecting the stochastic parts of the zero-point field [10–13].

In Section 3 we contrast the well-known physics behind the double slit with an emergent vector field representation of the observed interference field. The essential parts of our superclassical approach are presented and the velocity field corresponding to the guiding equation of the de Broglie-Bohm theory is derived. The explanation and validity of Born’s rule is analyzed in Section 4 by means of a three slit configuration. In Section 5 we summarize our results and give an outlook on possible limits of the validity of present-day quantum mechanics from the perspective of our sub-quantum approach.

2. INTRODUCING “SUPERCLASSICALITY” AND “RELATIONAL CAUSALITY”

In quantum mechanics, as well as in our quantum-like modeling via an emergent quantum mechanics approach, one can write down a formula for the total intensity distribution P which is identical to (1.3). For the general case of n slits, it holds with phase differences $\varphi_{ij} = \varphi_i - \varphi_j$ that

$$P = \sum_{i=1}^n \left(P_i + \sum_{j=i+1}^n 2R_iR_j \cos \varphi_{ij} \right), \quad (2.1)$$

where the phase differences are defined over the whole domain of the experimental setup. Apart from the role of the relative phase with important implications for the discussions on nonlocality [13], there is one additional ingredient that distinguishes (2.1) from its classical counterpart (1.3), namely the “dispersion of the wavepacket”. As in our model the “particle” is actually a “bouncer” in a fluctuating wave-like environment, i.e. analogously to the bouncers of Couder and Fort’s group, one does have some (e.g. Gaussian) distribution, with its center following the Ehrenfest trajectory in the free case, but one also has a diffusion to the right and to the left of the mean path which is just due to that stochastic bouncing. Thus the total velocity field of our bouncer in its fluctuating environment is given by the sum of the forward velocity \mathbf{v} and the respective diffusive velocities \mathbf{u}_L and \mathbf{u}_R to the left and the right. As for any direction i the diffusion velocity $\mathbf{u}_i = D \frac{\nabla_i P}{P}$ does not necessarily fall off with the distance, one has long effective tails of the distributions which contribute to the nonlocal nature of the interference phenomena [13]. In sum, one has essentially three velocity (or current) channels per slit in an n -slit system.

Earlier we showed that this phenomenon can also be understood as a variant of anomalous diffusion termed “ballistic diffusion”: a Brownian-type displacement with a time-dependent diffusivity (e.g., $D_t = \frac{D^2}{\sigma_0^2} t$ in the case of a Gaussian wave packet with standard deviation σ), leading to a “classically” obtained total velocity field

$$\mathbf{v}_{\text{tot}}(t) = \mathbf{v}(t) + [\mathbf{x}_{\text{tot}}(t) - \mathbf{v}(t)t] \frac{D_t}{\sigma^2}, \quad (2.2)$$

which is very practical to use in a computer simulation tool [14, 15]. Moreover, ballistic diffusion is a signature of what has in recent years become known as “superstatistics” [16]. Actually, the prototype of a phenomenon amenable to superstatistics is just a Brownian particle moving through a thermally changing environment: Combining in one formalism a relatively fast dynamics (e.g. particle velocity) on a small scale and slow changes (e.g. due to temperature fluctuations) on a large scale leads to a superposition of two statistics, i.e. superstatistics. One of the main features of superstatistics consists in *emergent properties arising on intermediate scales*, which may be *completely unexpected* if one looks only at either the small or the large scale.

The phenomenon just described, however, also provides us with the opportunity of solving a terminological problem with regard to sub-quantum theories such as ours. For, although we consider our approach as based on modern, “21st century classical physics”, the latter

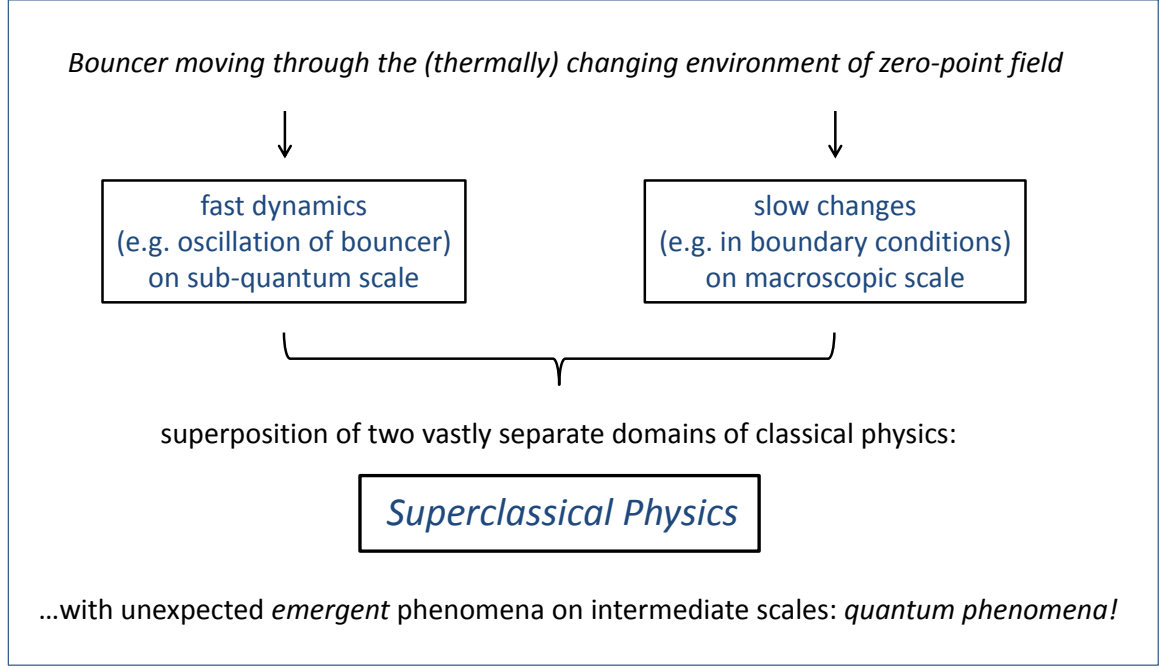


Figure 2.1: Scheme of *Superclassical Physics* as applied to our bouncer model

must not be confused with the term “classical physics” as referring to the state-of-art of, say, the early 20th century, including e.g. the time of the first experimental evidence of a zero-point energy (viz. Mulliken’s work from 1924 [17], which itself was before the advent of quantum theory proper). Acknowledging both that a) a zero-point field can in principle considered a “classical” one and that b) quantum theory is characterized by some decisively unexpected features when considered from a classical viewpoint, the phenomenon of superstatistics thus suggests the following analogy for the modeling of quantum systems: quantum behavior can be seen to emerge from the interplay of classical processes at very small (“sub-quantum”) and large (macroscopic) scales. A *combination of the physics on these vastly different scales* we call “superclassical” physics. The whole system under study then is characterized by *processes of emergence through the co-evolution of microscopic, local processes* (like the oscillations of a bouncer) *and of macroscopic processes* (like the time evolution of the experimental boundary conditions). In other words, starting from physical processes at such two vastly different classical scales, their superposition (i.e. within the framework of superclassical physics) makes possible new types of phenomena due to *emergent features unexpected from either the very small or the large scale physics* (Fig. 2.1; compare also the related concept of “emergent relativity” as discussed, e.g., by Jizba and Scardigli, this vol-

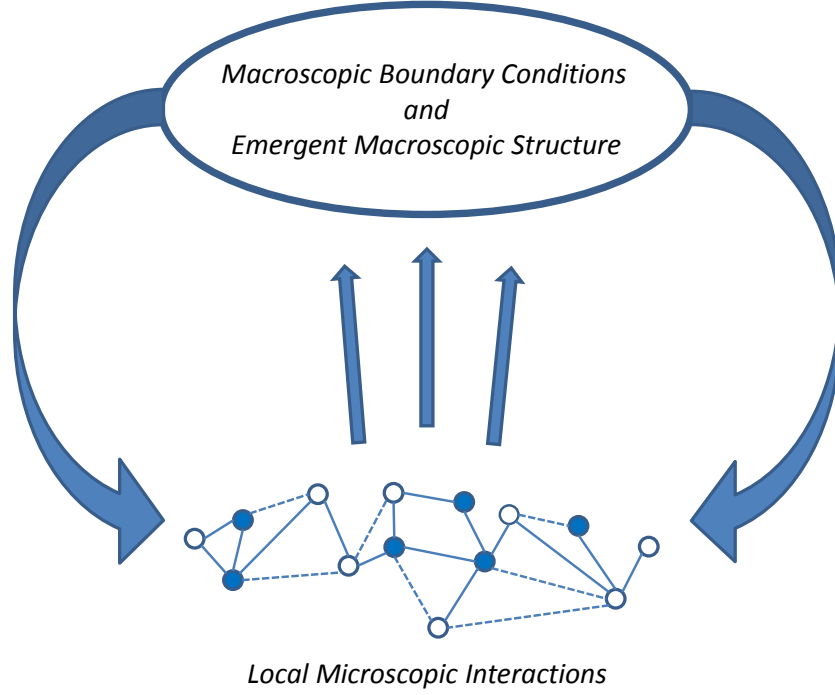


Figure 2.2: Scheme of *Relational Causality*: Co-evolution of bottom-up and top-down processes (after [20])

ume, [18], and [19]). In our development of an emergent quantum mechanics, we shall thus further on describe it as a superclassical approach, in order to avoid confusions when using “classical” explanations.

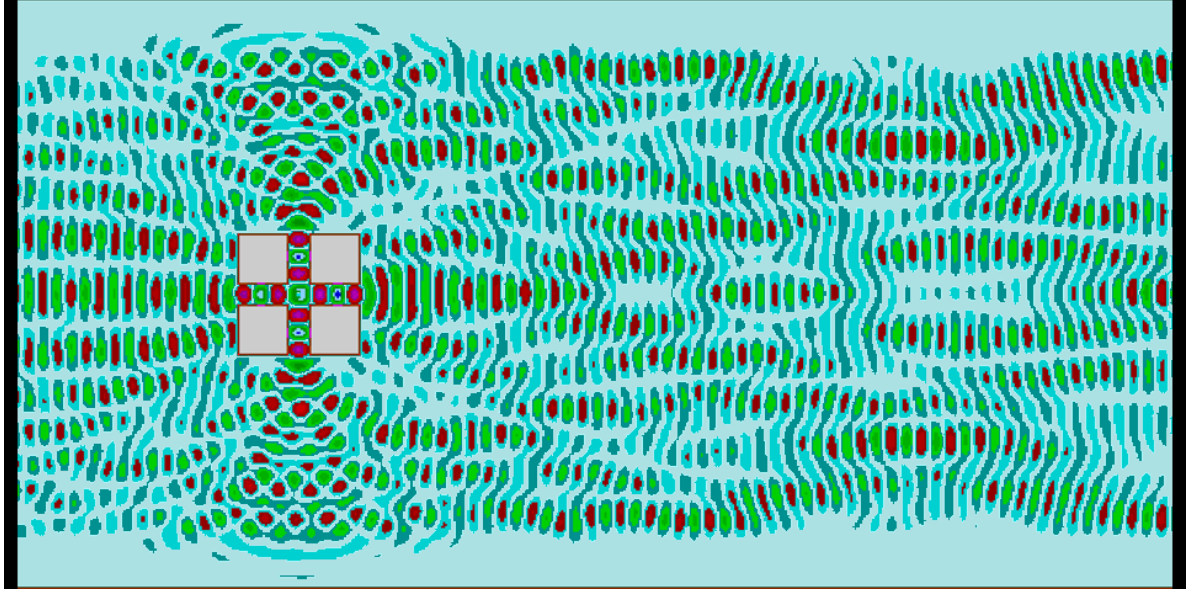
As mentioned, our superclassical approach is particularly suited to account for the emergent processes involved, i.e. for processes of a type that is well-known, for example, from the physics of Rayleigh-Bénard cells. The latter appear in a fluid subjected to a temperature gradient (like its container being heated from below) producing convection rolls, with the emergent particle trajectories strongly depending on the boundary conditions: The form of the convection rolls can be radically changed by changing the boundaries of the container. Generally, emergence of this type is characterized by *relational causality*, i.e. the co-evolution of bottom-up and top-down processes (Fig. 2.2).

In our emergent quantum mechanics approach, we have given a superclassical explanation of interference effects at a double slit [12], thereby arriving at expressions for the probability density current and the corresponding velocity field completely equivalent to the *guidance equation*, which is the central postulate of the de Broglie-Bohm interpretation [21, 22]. Our

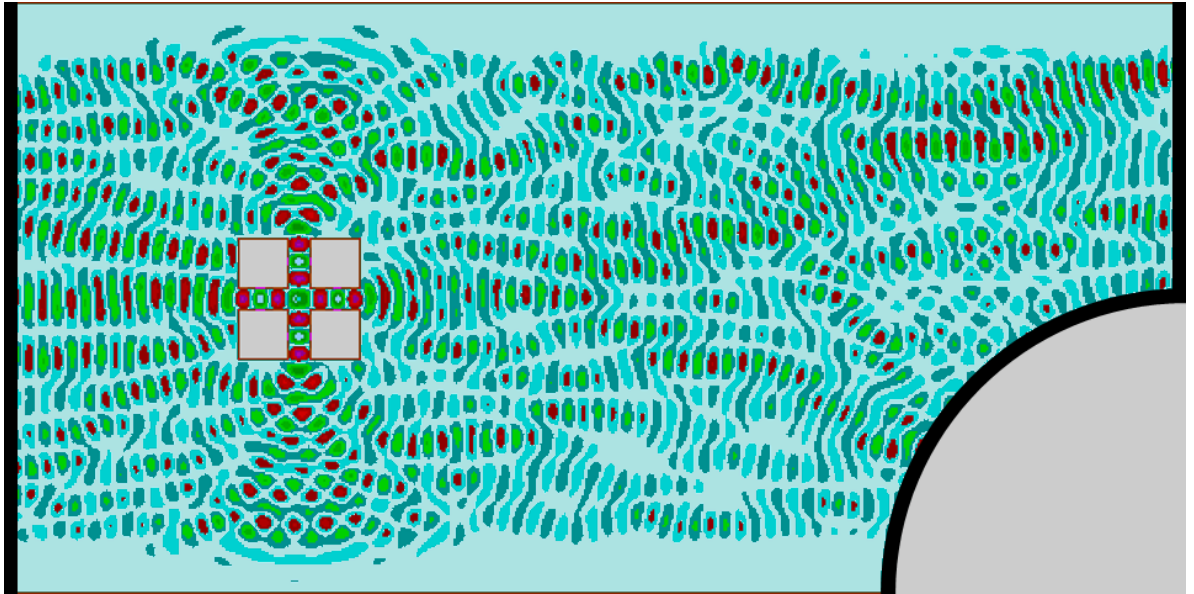
claims, to be substantiated in this paper, consist in the assertion that the guidance equation is of some “invisible hand” type, i.e. somehow mysteriously reaching out from configuration space in order to guide particles in real three-space. Instead, we shall argue, the guidance equation is completely understandable in real coordinate space, once the concept of emergence is taken seriously and introduced within a superclassical approach.

Note that a basic characteristic of emergent systems can be described via the above-mentioned relational causality (for a similar view, see [23]). Consider for example the following computer simulations (Figs. 2.3a and 2.3b). Similar to the experiments with droplets by Couder and Fort’s group, we insert onto a fluid surface a droplet in the center of the square constituted by four smaller squares of solid material, such that due to microscopic fluctuations the developing bouncer/walker will propagate along one of the four narrow paths until it finally escapes from the region of the square through one of the four slits into the open area. With the bouncer creating waves that will be reflected from the walls (drawn in black), after some time the whole system will develop standing waves between said walls, which act as physically effective boundary conditions. In Fig. 2.3a the walls act as simple plane mirrors, whereas in Fig. 2.3b the reflecting part on the right contains a circular structure, with the effect that the pattern of standing waves is altered. The figures display intensity distributions of the wave field which, accordingly, coincide with probability density distributions of finding the droplet at a specific location. Note that although the local physics in the vicinity of the square would be identical for all four particle trajectories emerging from one of the four slits *if* one disregarded the context, the pattern of the probability density field is different in both cases. This is of course due to the fact that one cannot, in principle, neglect the context, i.e. the boundary conditions create different standing wave patterns which in turn interfere with the otherwise identical local processes. *Relational causality* means that one must consider the *co-evolution of processes stemming from the local slits and those from the global environment*, the latter including the macroscopic boundary conditions. One can thus also speak of a “confluence” of different currents, i.e. those pertaining to the four slits and those coming from the larger environment. Therefore, if one records the bouncers’ trajectories and collects them in a synoptic manner, the whole velocity field will be a decisively *emergent* one. It is the totality of the whole experimental arrangement, and not just the classically local influences, that results in the behavior of the particle trajectories.

From this perspective, it is not very surprising that any relic of context-free modeling



(a) Illustration of *relational causality* under varying boundary conditions as described in the text, with standing waves between plane mirrors (black).



(b) Same as (a), with modified standing waves due to modified geometry of mirrors. Note the changes in the wave patterns around the squares despite the fact that the local physics there is kept unchanged. These changes are solely due to the altered boundary conditions.

Figure 2.3

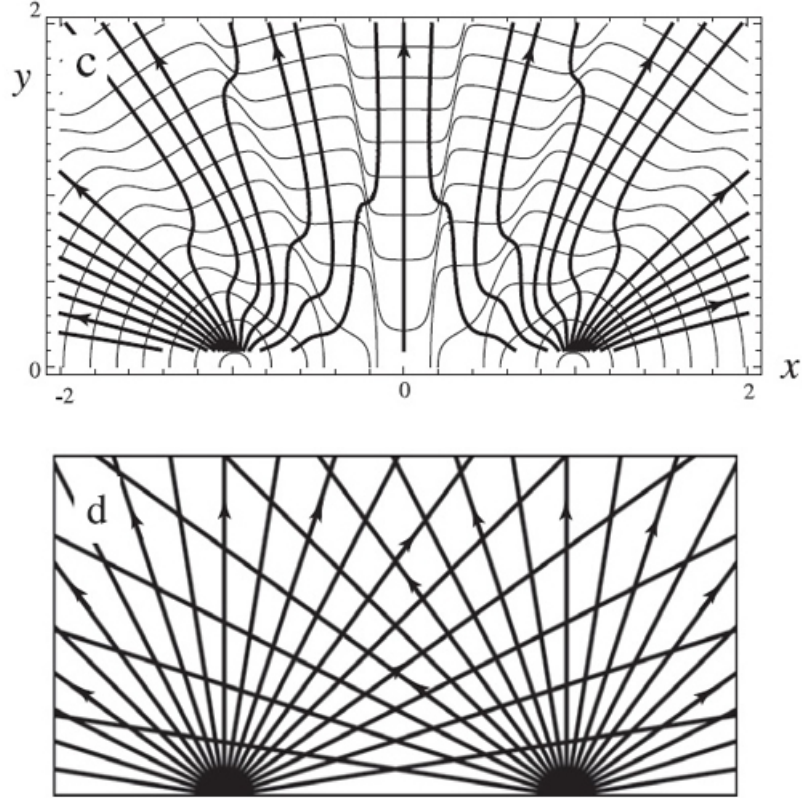


Figure 2.4: Quantum case streamlines (top) vs. geometrical rays (bottom). From [24]. Here it is demonstrated that by bringing together two sources, the temporal development of the velocity field (and the corresponding “trajectories”) in the quantum case can be understood only if considered as the result of a complex interplay of influences, updated at each instant of the time evolution, i.e. as the result of emergent processes.

should run into difficulties. Ironically, it was an orthodox position that has argued against the consequences of the more holistic approach acknowledging the context, i.e. when papers on apparently “surreal trajectories” were published against the assertion of the de Broglie-Bohm model that particles would not just follow trajectories as expected from context-free propagation obeying simple classically-local momentum conservation (see, e.g., the assertions of Scully [25]). The latter would apply only to geometrical rays, whereas the quantum case streamlines are more complicated, a situation well-known from optics (see Fig. 2.4 for illustration), and recently also confirmed via weak measurements in a double-slit experiment involving photons [26].

Having thus introduced the notions of superclassical physics and relational causality, we

are now ready to reconsider interference at the double slit. Recall that to arrive at the *classical* double slit formula for the intensities on the screen one had to just consider the mean velocity \mathbf{v} , i.e. there was just one channel (of the velocity, or the probability density current, respectively) per slit. Because the currents represent wave propagations, this has led to the total intensity of (1.3), $P = P(1) + P(2) = R_1^2 + R_2^2 + 2R_1R_2 \cos \varphi$. Now, however, we are going to look for the *superclassical* interference formula. As mentioned, we now have to deal with three channels per slit, due to the additional sub-quantum diffusion velocities \mathbf{u}_L and \mathbf{u}_R , next to \mathbf{v} , and *all three co-evolving*. Surprisingly, this is all that is needed to arrive at the well-known quantum mechanical results which usually are obtained only via complex-valued probability amplitudes.

3. A SUPERCLASSICAL DERIVATION OF THE GUIDANCE EQUATION

Considering particles as oscillators (“bouncers”) coupling to regular oscillations of the vacuum’s zero-point field, which they also generate, we have shown how a quantum can be understood as an emergent system. In particular, with the dynamics between the oscillator on the one hand, and the “bath” of its thermal environment as constrained by the experiment’s boundary conditions on the other hand, one can explain not only Gaussian diffraction at a single slit [15], but also the well-known interference effects at a double slit [12, 27]. As already mentioned, we have also shown that the spreading of a wave packet can be exactly described by combining the convective with the orthogonal diffusive velocity fields. The latter fulfill the condition of being unbiased w.r.t. the convective velocities, i.e. the orthogonality relation for the *averaged* velocities derived in [15] is $\overline{\mathbf{v}\mathbf{u}} = 0$, since any fluctuations $\mathbf{u} = \delta(\nabla S/m)$ are shifts along the surfaces of action $S = \text{const.}$

To account for the different velocity channels $i = 1, \dots, 3n$, n being the number of slits, we now introduce for general cases the generalized velocity vectors \mathbf{w}_i , which in the case of $n = 2$ are

$$\mathbf{w}_1 := \mathbf{v}_1, \quad \mathbf{w}_2 := \mathbf{u}_{1R}, \quad \mathbf{w}_3 := \mathbf{u}_{1L} \quad (3.1)$$

for the first channel, and

$$\mathbf{w}_4 := \mathbf{v}_2, \quad \mathbf{w}_5 := \mathbf{u}_{2R}, \quad \mathbf{w}_6 := \mathbf{u}_{2L} \quad (3.2)$$

for the second channel. The associated amplitudes $R(\mathbf{w}_i)$ for each channel are taken to be

the same, i.e. $R(\mathbf{w}_1) = R(\mathbf{w}_2) = R(\mathbf{w}_3) = R_1$, and $R(\mathbf{w}_4) = R(\mathbf{w}_5) = R(\mathbf{w}_6) = R_2$.

Now, relational causality manifests itself in that the total wave intensity field consists of the sum of all local intensities in each channel, and the local intensity in each channel is the result of the interference with the total intensity field. Thus, any change in the local field affects the total field, and *vice versa*: Any change in the total field affects the local one. In order to completely accommodate the totality of the system, we therefore need to define a “wholeness”-related local wave intensity $P(\mathbf{w}_i)$ in one channel (i.e. \mathbf{w}_i) upon the condition that the totality of the superposing waves is given by the “rest” of the $3n - 1$ channels. Concretely, we account for the phase-dependent amplitude contributions of the total system’s wave field projected on one channel’s amplitude $R(\mathbf{w}_i)$ at the point (\mathbf{x}, t) with a *conditional probability density* $P(\mathbf{w}_i)$. The expression for $P(\mathbf{w}_i)$ thus constitutes the representation of relational causality within our ansatz. Moreover, as usual one can define a local current $\mathbf{J}(\mathbf{w}_i)$ per channel as the corresponding “local” intensity-weighted velocity \mathbf{w}_i . Since the two-path set-up has $3n = 6$ velocity vectors at each point (cf. Eqs. (3.1) and (3.2)), we thus obtain for the partial intensities and currents, respectively, i.e. for each channel component i ,

$$P(\mathbf{w}_i) = R(\mathbf{w}_i) \hat{\mathbf{w}}_i \cdot \sum_{j=1}^6 \hat{\mathbf{w}}_j R(\mathbf{w}_j) \quad (3.3)$$

$$\mathbf{J}(\mathbf{w}_i) = \mathbf{w}_i P(\mathbf{w}_i), \quad i = 1, \dots, 6, \quad (3.4)$$

with

$$\cos \varphi_{i,j} := \hat{\mathbf{w}}_i \cdot \hat{\mathbf{w}}_j. \quad (3.5)$$

Consequently, the total intensity and current of our field read as

$$P_{\text{tot}} = \sum_{i=1}^6 P(\mathbf{w}_i) = \left(\sum_{i=1}^6 \hat{\mathbf{w}}_i R(\mathbf{w}_i) \right)^2 \quad (3.6)$$

$$\mathbf{J}_{\text{tot}} = \sum_{i=1}^6 \mathbf{J}(\mathbf{w}_i) = \sum_{i=1}^6 \mathbf{w}_i P(\mathbf{w}_i), \quad (3.7)$$

leading to the *emergent total velocity*

$$\mathbf{v}_{\text{tot}} = \frac{\mathbf{J}_{\text{tot}}}{P_{\text{tot}}} = \frac{\sum_{i=1}^6 \mathbf{w}_i P(\mathbf{w}_i)}{\sum_{i=1}^6 P(\mathbf{w}_i)}. \quad (3.8)$$

Returning now to our previous notation for the six velocity components \mathbf{v}_i , \mathbf{u}_{iR} , \mathbf{u}_{iL} , $i = 1, 2$, the partial current associated with \mathbf{v}_1 originates from building the scalar product of $\hat{\mathbf{v}}_1$ with all other unit vector components and reads as

$$\mathbf{J}(\mathbf{v}_1) = \mathbf{v}_1 P(\mathbf{v}_1) = \mathbf{v}_1 R_1 \hat{\mathbf{v}}_1 \cdot (\hat{\mathbf{v}}_1 R_1 + \hat{\mathbf{u}}_{1R} R_1 + \hat{\mathbf{u}}_{1L} R_1 + \hat{\mathbf{v}}_2 R_2 + \hat{\mathbf{u}}_{2R} R_2 + \hat{\mathbf{u}}_{2L} R_2). \quad (3.9)$$

Since trivially

$$\hat{\mathbf{u}}_{iR} R_i + \hat{\mathbf{u}}_{iL} R_i = 0, \quad i = 1, 2, \quad (3.10)$$

Eq. (3.9) leads to

$$\mathbf{J}(\mathbf{v}_1) = \mathbf{v}_1 (R_1^2 + R_1 R_2 \cos \varphi), \quad (3.11)$$

which results from the representation of the emerging velocity fields, since we get the cosine of the phase difference φ as a natural result of the scalar product of the velocity vectors \mathbf{v}_i . The non-zero residua of the other vector fields yield

$$\mathbf{J}(\mathbf{u}_{1R}) = u_{1R} P(\mathbf{u}_{1R}) = \mathbf{u}_{1R} (R_1 \hat{\mathbf{u}}_{1R} \cdot \hat{\mathbf{v}}_2 R_2) = \mathbf{u}_{1R} R_1 R_2 \cos \left(\frac{\pi}{2} - \varphi \right) = \mathbf{u}_{1R} R_1 R_2 \sin \varphi \quad (3.12)$$

and

$$\mathbf{J}(\mathbf{u}_{1L}) = \mathbf{u}_{1L} P(\mathbf{u}_{1L}) = \mathbf{u}_{1L} (R_1 \hat{\mathbf{u}}_{1L} \cdot \hat{\mathbf{v}}_2 R_2) = \mathbf{u}_{1L} R_1 R_2 \cos \left(\frac{\pi}{2} + \varphi \right) = -\mathbf{u}_{1L} R_1 R_2 \sin \varphi. \quad (3.13)$$

Analogously, we obtain for the convective velocity vector field of the second channel

$$\mathbf{J}(\mathbf{v}_2) = \mathbf{v}_2 P(\mathbf{v}_2) = \mathbf{v}_2 (R_2^2 + R_1 R_2 \cos \varphi). \quad (3.14)$$

The corresponding diffusive velocity vector fields read as

$$\mathbf{J}(\mathbf{u}_{2R}) = \mathbf{u}_{2R} P(\mathbf{u}_{2R}) = \mathbf{u}_{2R} (R_2 \hat{\mathbf{u}}_{2R} \cdot \hat{\mathbf{v}}_1 R_1) = \mathbf{u}_{2R} R_1 R_2 \cos \left(\frac{\pi}{2} + \varphi \right) = -\mathbf{u}_{2R} R_1 R_2 \sin \varphi, \quad (3.15)$$

$$\mathbf{J}(\mathbf{u}_{2L}) = \mathbf{u}_{2L} P(\mathbf{u}_{2L}) = \mathbf{u}_{2L} (R_2 \hat{\mathbf{u}}_{2L} \cdot \hat{\mathbf{v}}_1 R_1) = \mathbf{u}_{2L} R_1 R_2 \cos \left(\frac{\pi}{2} - \varphi \right) = \mathbf{u}_{2L} R_1 R_2 \sin \varphi. \quad (3.16)$$

Note that the nontrivial sine contributions to the total current stem from the projections between the diffusive velocities $\mathbf{u}_{1R(L)}$ of the first channel on the unit vector $\hat{\mathbf{v}}_2$ of the

convective velocity of the second channel, and *vice versa*. Combining all terms, we obtain with Eq. (3.7) the result for the total current

$$\begin{aligned}\mathbf{J}_{\text{tot}} &= \mathbf{v}_1 P(\mathbf{v}_1) + \mathbf{u}_{1R} P(\mathbf{u}_{1R}) + \mathbf{u}_{1L} P(\mathbf{u}_{1L}) + \mathbf{v}_2 P(\mathbf{v}_2) + \mathbf{u}_{2R} P(\mathbf{u}_{2R}) + \mathbf{u}_{2L} P(\mathbf{u}_{2L}) \\ &= R_1^2 \mathbf{v}_1 + R_2^2 \mathbf{v}_2 + R_1 R_2 (\mathbf{v}_1 + \mathbf{v}_2) \cos \varphi + R_1 R_2 ([\mathbf{u}_{1R} - \mathbf{u}_{1L}] - [\mathbf{u}_{2R} - \mathbf{u}_{2L}]) \sin \varphi.\end{aligned}\quad (3.17)$$

The resulting diffusive velocities $\mathbf{u}_{iR} - \mathbf{u}_{iL}$ are identified with the effective diffusive velocities \mathbf{u}_i for each channel. Note that *one* of those velocities, \mathbf{u}_{iR} or \mathbf{u}_{iL} , respectively, is always zero, so that the product of said difference with $\sin \varphi$ guarantees the correct sign of the last term in Eq. (3.17). Thus we obtain the final expression for the total density current built from the remaining $2n = 4$ velocity components

$$\mathbf{J}_{\text{tot}} = R_1^2 \mathbf{v}_1 + R_2^2 \mathbf{v}_2 + R_1 R_2 (\mathbf{v}_1 + \mathbf{v}_2) \cos \varphi + R_1 R_2 (\mathbf{u}_1 - \mathbf{u}_2) \sin \varphi. \quad (3.18)$$

The obtained total density current field $\mathbf{J}_{\text{tot}}(\mathbf{x}, t)$ spanned by the various velocity components $\mathbf{v}_i(\mathbf{x}, t)$ and $\mathbf{u}_{iR(L)}(\mathbf{x}, t)$ we have denoted as the “path excitation field” [12]. It is built by the sum of all its partial currents, which themselves are built by an amplitude weighted projection of the total current.

Summing up the probabilities associated with each of the partial currents we obtain according to the ansatz (3.3) and the relations (3.6) and (3.10)

$$\begin{aligned}P_{\text{tot}} &= (R_1 \hat{\mathbf{v}}_1 + R_1 \hat{\mathbf{u}}_{1R} + R_1 \hat{\mathbf{u}}_{1L} + R_2 \hat{\mathbf{v}}_2 + R_2 \hat{\mathbf{u}}_{2R} + R_2 \hat{\mathbf{u}}_{2L})^2 \\ &= (R_1 \hat{\mathbf{v}}_1 + R_2 \hat{\mathbf{v}}_2)^2 = R_1^2 + R_2^2 + 2R_1 R_2 \cos \varphi = P(\mathbf{v}_1) + P(\mathbf{v}_2).\end{aligned}\quad (3.19)$$

The total velocity \mathbf{v}_{tot} according to Eq. (3.8) now reads as

$$\mathbf{v}_{\text{tot}} = \frac{R_1^2 \mathbf{v}_1 + R_2^2 \mathbf{v}_2 + R_1 R_2 (\mathbf{v}_1 + \mathbf{v}_2) \cos \varphi + R_1 R_2 (\mathbf{u}_1 - \mathbf{u}_2) \sin \varphi}{R_1^2 + R_2^2 + 2R_1 R_2 \cos \varphi}. \quad (3.20)$$

The trajectories or streamlines, respectively, are obtained according to $\dot{\mathbf{x}} = \mathbf{v}_{\text{tot}}$ in the usual way by integration. As first shown in [12], by re-inserting the expressions for convective and diffusive velocities, respectively, i.e. $\mathbf{v}_{i,\text{conv}} = \frac{\nabla S_i}{m}$, $\mathbf{u}_i = -\frac{\hbar}{m} \frac{\nabla R_i}{R_i}$, one immediately identifies Eq. (3.20) with the Bohmian guidance equation and Eq. (3.18) with the quantum mechanical pendant for the probability density current [28]. The latter can be seen as follows. Upon employment of the Madelung transformation for each path j ($j = 1$ or 2),

$$\psi_j = R_j e^{iS_j/\hbar}, \quad (3.21)$$

and thus $P_j = R_j^2 = |\psi_j|^2 = \psi_j^* \psi_j$, with $\varphi = (S_1 - S_2)/\hbar$, and recalling the usual trigonometric identities such as $\cos \varphi = \frac{1}{2} (e^{i\varphi} + e^{-i\varphi})$, one can rewrite the total average current (3.18) immediately as

$$\begin{aligned} \mathbf{J}_{\text{tot}} &= P_{\text{tot}} \mathbf{v}_{\text{tot}} \\ &= (\psi_1 + \psi_2)^* (\psi_1 + \psi_2) \frac{1}{2} \left[\frac{1}{m} \left(-i\hbar \frac{\nabla(\psi_1 + \psi_2)}{(\psi_1 + \psi_2)} \right) + \frac{1}{m} \left(i\hbar \frac{\nabla(\psi_1 + \psi_2)^*}{(\psi_1 + \psi_2)^*} \right) \right] \quad (3.22) \\ &= -\frac{i\hbar}{2m} [\Psi^* \nabla \Psi - \Psi \nabla \Psi^*] = \frac{1}{m} \text{Re} \{ \Psi^* (-i\hbar \nabla) \Psi \}, \end{aligned}$$

where $P_{\text{tot}} = |\psi_1 + \psi_2|^2 =: |\Psi|^2$. The last two expressions of (3.22) are the exact formulations of the quantum mechanical probability current, here obtained just by a re-formulation of (3.18). In fact, it is a simple exercise to insert the wave functions (3.21) into (3.22) to re-obtain (3.18).

Note that it is straightforward to extend this derivation to the many-particle case. As the individual terms in the expressions for the total current and total probability density, respectively, are purely additive also for N particles, a fact that is well-known also from Bohmian theory, the above-mentioned “translation” into orthodox quantum language is straightforward, with the effect that the currents’ nabla operators just have to be applied at all of the locations \mathbf{x} of the respective N particles, thus providing the quantum mechanical formula $\mathbf{J}_{\text{tot}}(N) = \frac{1}{m} \text{Re} \{ \Psi^* (-i\hbar \nabla_N) \Psi \}$, where Ψ now is the total N -particle wave function.

Again we emphasize that our result was obtained solely out of kinematic relations by applying the superclassical rules introduced above on the basis of a relational causality, i.e. without invoking complex ψ functions or the like. Moreover, as opposed to the Bohmian theory, we obtained our results not in configuration space but in ordinary coordinate space. *What looks like the necessity to superpose wave functions in configuration space, which then are imagined to guide the particles by some invisible hand, can equally be obtained by superpositions of all relational amplitude configurations of waves in real space, i.e. by understanding the resulting system’s evolutions as processes of emergence.*

Thus, with $\mathbf{w}_i = \frac{\mathbf{J}(\mathbf{w}_i)}{P(\mathbf{w}_i)}$ and the classical composition principles of Eqs. (3.6) and (3.7) we have shown that the total velocity field is given in the simple form of a *(super)classical average velocity field*:

$$\mathbf{v}_{\text{tot}} = \frac{\mathbf{J}_{\text{tot}}}{P_{\text{tot}}} = \frac{\sum_i \mathbf{J}(\mathbf{w}_i)}{\sum_i P(\mathbf{w}_i)} = \frac{\sum_i \mathbf{w}_i P(\mathbf{w}_i)}{\sum_i P(\mathbf{w}_i)}. \quad (3.23)$$

In other words, the guidance equation postulated by de Broglie-Bohm is here derived and explained via relational causality, with \mathbf{v}_{tot} being an *emergent velocity field*.

4. THREE-SLIT INTERFERENCE, BORN'S RULE, AND SORKIN'S SUM RULES

The extension to three slits, beams, or probability current channels, respectively, is straightforward. We just introduce a third emergent propagation velocity \mathbf{v}_3 and its corresponding diffusive velocities $\mathbf{u}_{3L(R)}$. The phase shift of the third beam is denoted as χ and represents the angle between the second and the third beam in our geometric representation of the path excitation field. According to Born's rule the probability of even a single particle passing any of the three slits splits into a sum of probabilities passing the slits pairwise, i.e. going along both A and B , B and C , or A and C , but never passing A , B and C simultaneously.

Interference phenomena have recently been analyzed thoroughly for the cases of only one open slit up to n open slits by Sorkin [29]. For a double slit setup the interference term is non-zero, i.e. $I_{AB} := P_{AB} - P_A - P_B \neq 0$, with $P_{A(B)}$ being the detection probability with only one slit/path A or B , respectively, of a total of n slits/paths open, and P_{AB} for both slits A and B open. This “first order sum rule” is to be contrasted with Sorkin's results for the following, so-called “second order sum rule” [29]:

$$\begin{aligned} I_{ABC} &:= P_{ABC} - P_{AB} - P_{AC} - P_{BC} + P_A + P_B + P_C \\ &= P_{ABC} - (P_A + P_B + P_C + I_{AB} + I_{AC} + I_{BC}) = 0. \end{aligned} \quad (4.1)$$

This result is remarkable insofar as it can be inferred that interference terms theoretically always originate from pairings of paths, but never from triples etc. Any violation of this second order sum rule, i.e. $I_{ABC} \neq 0$, and thus of Born's rule would have dramatic consequences for quantum theory like a modification of the Schrödinger equation, for example.

Returning to our model, the total probability density current for three paths is calculated according to the rules set up in Section 3. We adopt the notations of the two slit system also for three slits, i.e. now employing nine velocity contributions: \mathbf{v}_i , $\mathbf{u}_{iR(L)}$, $i = 1, 2, 3$. Analogously, the three generally different amplitudes are denoted as $R(\mathbf{v}_i) = R(\mathbf{u}_{iR}) =$

$R(\mathbf{u}_{iL}) = R_i$, $i = 1, 2, 3$. We keep the definition of φ as $\varphi := \arccos(\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2)$, and we define the second angle as $\chi := \arccos(\hat{\mathbf{v}}_2 \cdot \hat{\mathbf{v}}_3)$. Similarly to Eq. (3.10), the diffusive velocities $\mathbf{u}_{iR} - \mathbf{u}_{iL}$ combine to \mathbf{u}_i , $i = 1, 2, 3$, thus ending up with $2n = 6$ effective velocities. Therefore we obtain, analogously to the calculation in the previous section,

$$\begin{aligned} \mathbf{J}_{\text{tot}} = & R_1^2 \mathbf{v}_1 + R_2^2 \mathbf{v}_2 + R_3^2 \mathbf{v}_3 + R_1 R_2 (\mathbf{v}_1 + \mathbf{v}_2) \cos \varphi + R_1 R_2 (\mathbf{u}_1 - \mathbf{u}_2) \sin \varphi \\ & + R_1 R_3 (\mathbf{v}_1 + \mathbf{v}_3) \cos (\varphi + \chi) + R_1 R_3 (\mathbf{u}_1 - \mathbf{u}_3) \sin (\varphi + \chi) \\ & + R_2 R_3 (\mathbf{v}_2 + \mathbf{v}_3) \cos \chi + R_2 R_3 (\mathbf{u}_2 - \mathbf{u}_3) \sin \chi \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} P_{\text{tot}} = & R_1^2 + R_2^2 + R_3^2 + 2R_1 R_2 \cos \varphi + 2R_1 R_3 \cos (\varphi + \chi) + 2R_2 R_3 \cos \chi \\ = & P(\mathbf{v}_1) + P(\mathbf{v}_2) + P(\mathbf{v}_3). \end{aligned} \quad (4.3)$$

In analogy to the double slit case (cf. Eq. (3.19)) we obtain a *classical* Kolmogorov sum rule for the probabilities on the one hand, but also the complete interference effects for the double, three- and, as we have shown in [27], for the n -slit cases, on the other. However, the particular probabilities $P(\mathbf{v}_i)$ in Eqs. (3.19) and (4.3), do not correspond to the probabilities of the assigned slits if solely opened, i.e. $P_{AB}(\mathbf{v}_1) = (R_1^2 + R_1 R_2 \cos \varphi) \neq P_A(\mathbf{v}_1) = R_1^2$. Consequently, each of the probability summands in said equations does *not* correspond to an independent probability of the respective slit if solely opened, a fact that was already clarified in our discussion of the issue of contexts in Section ?? .

Finally, we obtain for the cases of one (i.e. $n = A$), two and three open slits, respectively,

$$I_A = P_A(\mathbf{v}_1) = R_1^2, \quad (4.4)$$

$$I_{AB} = P_{AB} - P_A(\mathbf{v}_1) - P_B(\mathbf{v}_2) = 2R_1 R_2 \cos \varphi, \quad (4.5)$$

$$I_{ABC} = P_{ABC} - P_{AB} - P_{AC} - P_{BC} + P_A(\mathbf{v}_1) + P_B(\mathbf{v}_2) + P_C(\mathbf{v}_3) = 0, \quad (4.6)$$

where P_{AB} is assigned to P_{tot} of Eq. (3.19) and P_{ABC} to P_{tot} of Eq. (4.3). In the double slit case, e.g., with slits A and B open, we obtain the results of (3.19). If B were closed and C were open instead, we would get the analogous result, i.e. \mathbf{v}_2 and φ replaced by \mathbf{v}_3 and $\varphi_{1,3}$. If all three slits A, B, C are open, we can use the pairwise permutations of the double slit case, i.e. $A \wedge B$, $A \wedge C$, or $B \wedge C$, respectively, with $\varphi_{1,3}$ identified with $(\varphi + \chi)$, etc.

Thus we conclude that in our model the addition of “sub-probabilities” indeed works and provides the correct results.

Summarizing, with our superclassical model emerging out of a sub-quantum scenario we arrive at the same results as standard quantum mechanics fulfilling Sorkin’s sum rules [29]. However, whereas in standard quantum mechanics Born’s rule originates from building the squared absolute values of additive ψ functions representing the probability amplitudes for different paths, in our case we obtain the pairing of paths as a natural consequence of the *pairwise* selection of unit vectors of all existing velocity components constituting the probability currents. Thus we obtain *all* possible pathways within an n -slit setup by our projection method. The sum rules, Eqs. (3.3) through (3.8), guarantee that each partial contribution, be it from the velocity contributions within a particular channel or from different channels, accounts for the final total current density for each point between source and detector. Since for only one slit open the projection rule (3.3) trivially leads to a linear relation between P and R^2 , the asymmetry between the latter quantities, due to the nonlinear projection rule, becomes effective for $n \geq 2$ slits open. Consequently, the violation of the first order sum rule (4.5), i.e. $I_{AB} \neq 0$, represents a *natural* result of our principle of relational causality. Moreover, as we have argued above, the opening of an additional slit solely adds pairwise path combinations. As all higher interference terms have already incorporated said asymmetry, the result can finally be reduced to the double slit case, thus yielding a zero result as in Eq. (4.6) according to Sorkin’s analysis.

This is a further hint that our model can reproduce all phenomena of standard quantum theory with the option of giving a deeper reasoning to principles like Born’s rule or the hierarchical sum-rules, respectively.

5. CONCLUSIONS AND OUTLOOK

We have previously shown in a series of papers [10–15, 27, 30] that phenomena of standard quantum mechanics like Gaussian dispersion of wave packets, superposition, double slit interference, Planck’s energy relation, or the Schrödinger equation can be assessed as the emergent property of an underlying sub-structure of the vacuum combined with diffusion processes reflecting also the stochastic parts of the zero-point field, i.e. the zero point fluctuations. (For similar approaches see the works of Cetto and de la Peña [31, and this

volume], Nieuwenhuizen [this volume], or Khrennikov *et al.* [32].) Thus we obtain the quantum mechanical dynamics as an averaged behavior of sub-quantum processes. The inclusion of relativistic physics has not been considered yet, but should be possible in principle.

By introducing the concepts of superclassicality and relational causality, we have in this paper shown that quantum phenomenology can be meaningfully grounded in a superclassical approach relying solely on classical probability theory. Apart from an application for a deeper understanding of Born's rule, the central result of this work is a demonstration that the guidance equation can be derived and explained within ordinary coordinate space. We have proven the identity of our emergent velocity field \mathbf{v}_{tot} with the corresponding Bohmian one, $\mathbf{v}_{\text{tot(Bohm)}}$, and the orthodox quantum mechanical one, $\mathbf{v}_{\text{tot(QM)}}$, respectively:

$$\begin{aligned}
\mathbf{v}_{\text{tot(emergent)}} &= \frac{\sum_i \mathbf{w}_i P(\mathbf{w}_i)}{\sum_i P(\mathbf{w}_i)} \\
&= \mathbf{v}_{\text{tot(Bohm)}} = \frac{R_1^2 \mathbf{v}_1 + R_2^2 \mathbf{v}_2 + R_1 R_2 (\mathbf{v}_1 + \mathbf{v}_2) \cos \varphi + R_1 R_2 (\mathbf{u}_1 - \mathbf{u}_2) \sin \varphi}{R_1^2 + R_2^2 + 2R_1 R_2 \cos \varphi} \quad (5.1) \\
&= \mathbf{v}_{\text{tot(QM)}} = \frac{1}{m|\Psi|^2} \text{Re} \{ \Psi^* (-i\hbar \nabla) \Psi \}, \quad \text{with } \Psi = \sum_j \psi_j.
\end{aligned}$$

Finally, with our superclassical theory we can also enquire into the possible limits of present-day quantum theory. For example, the latter is expected to break down at the time scales of our bouncer's oscillation frequency, e.g., for the electron $\omega \approx \mathcal{O}(10^{21} \text{ Hz})$. As we have seen, at the emerging quantum level, i.e. at times $t \gg 1/\omega$, we obtain exact results strongly suggesting the validity of Born's rule, for example. However, approaching said sub-quantum regions by increasing the time resolution to the order of $t \approx 1/\omega$ suggests a possibly gradual breakdown of said rule, since the averaging of the diffusive and convective velocities and their mutual orthogonality of the averaged velocities is not reliable any more. In principle, this should eventually be testable in experiment. Moreover, upon the velocities \mathbf{v} and $\mathbf{u}_{\text{L(R)}}$, introduction of a new bias, either in the average orthogonality condition, or between the different velocity channels, the question may be of relevance whether these would lead to the collapse of the superposition principle, as the assumed sub-quantum nonlinearities would then become manifest. We have not discussed the important issue of

nonlocality and possible consequences with regard to the non-signaling principle here, and refer the reader to the paper by Jan Walleczek (this volume) for consideration of some of the topics in question.

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